Grupa A

1	2	3	4	$\Sum$

BROJ INDEKSA

SMJER STUDIJA

IME I PREZIME

Prilikom pisanja rješenja zadataka obratiti pažnju na matematičku kulturu i matematičku pismenost

Matematika II, pismeni ispit, 08.10.2014.

1. Figura u ravni ograničena parabolom $y = 4 - x^2$ i poluravnima $y \geq x$, $y \geq 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

2. Naći ekstreme funkcije $z = x + y + 4 + 4 \sin x \sin y$.

3. Date su vrijednosti dva integrala ($\alpha > 0$)

$$\int_0^\infty \frac{\cos \alpha x}{1+x^2} \mathrm{d}x = \frac{\pi}{2} e^{-\alpha}, \quad \int_0^\infty \frac{\sin \alpha x}{x} \mathrm{d}x = \frac{\pi}{2}.$$

Koristeći date jednakosti, uz pomoć metode diferenciranja po parametru izračunati $\displaystyle\int_0^\infty \frac{\sin \alpha x}{x(1+x^2)} \mathrm{d}x$.

4. Naći fluks polja $\vec{v} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ kroz dio sfere $x^2 + y^2 + z^2 = 1$ u I oktantu.

VAŽNO: Ovaj papir treba predati zajedno s rješenjima zadataka! Ispit pisati isključivo hemijskom olovkom plave ili crne tinte.

Grupa B

1	2	3	4	$\Sum$

BROJ INDEKSA

SMJER STUDIJA

IME I PREZIME

Prilikom pisanja rješenja zadataka obratiti pažnju na matematičku kulturu i matematičku pismenost

Matematika II, pismeni ispit, 08.10.2014.

1. Figura u ravni ograničena linijama $2y = x^2$ i $2x + 2y - 3 = 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

2. Odrediti ekstreme funkcije $f(x, y) = xe^{y + x \sin y}$.

3. Prvo izračunati integral $I = \displaystyle\int_0^\infty e^{-x} \sin(\alpha x) \mathrm{d}x$ pa poslije toga dobijeni rezultat iskoristiti i koristeći metodu diferenciranja po parametru izračunati

$$G(\alpha) = \int_0^\infty xe^{-x} \cos(\alpha x) \mathrm{d}x$$

4. Izračunati tok (fluks) vektora $\vec{v} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ kroz sferu $x^2 + y^2 + z^2 = R^2$.

VAŽNO: Ovaj papir treba predati zajedno s rješenjima zadataka! Ispit pisati isključivo hemijskom olovkom plave ili crne tinte.

\# Figura u ravni ograničena parabolom $y=4-x^2$ i poluravnima $y \geq x$, $y \geq 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

1,56
-3,56

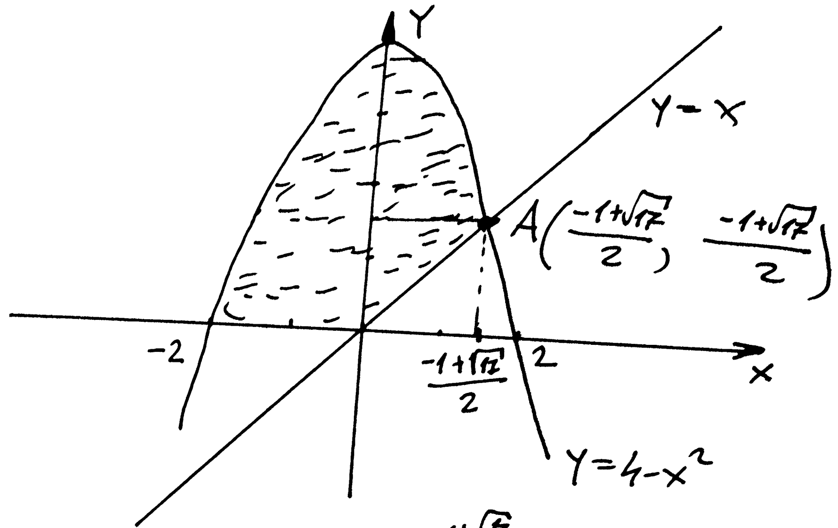
Rj.

$$\begin{aligned} y &= 4-x^2 \\ y &= x \\ \hline x &= 4-x^2 \\ x^2+x-4 &= 0 \\ D &= 1+16=17 \end{aligned}$$

$$x_{1,2} = \frac{-1 \pm \sqrt{17}}{2}$$

$$x_1 = \frac{-1-\sqrt{17}}{2}$$

$$x_2 = \frac{-1+\sqrt{17}}{2}$$



$$V_x = V_1 - V_2, \quad V_1 = \pi \int_{-2}^{\frac{-1+\sqrt{17}}{2}} (4-x^2)^2\,dx, \quad V_2 = \pi \int_0^{\frac{-1+\sqrt{17}}{2}} (x)^2\,dx$$

$$V_1 = \pi \int_{-2}^{\frac{-1+\sqrt{17}}{2}} (16-8x^2+x^4)\,dx = \pi \Bigg(\underbrace{16x \Big|_{-2}^{\frac{-1+\sqrt{17}}{2}}}_{16\left(\frac{-1+\sqrt{17}}{2}-(-2)\right)} - \frac{8}{3} \underbrace{x^3 \Big|_{-2}^{\frac{-1+\sqrt{17}}{2}}}_{\frac{(-1+\sqrt{17})^3}{8}-(-2)^3} + \frac{1}{5} \underbrace{x^5 \Big|_{-2}^{\frac{-1+\sqrt{17}}{2}}}_{\frac{(-1+\sqrt{17})^5}{2^5}-(-2)^5} \Bigg) =$$

$$= 8\sqrt{17} + 24 - \frac{\sqrt{17}^3}{3} + \sqrt{17}^2 - \sqrt{17} - 21 + \frac{1}{5} \cdot \left[\frac{(-1+\sqrt{17})^5}{2^5} + 2^5 \right] =$$

$$= -\frac{\sqrt{17}^3}{3} + 7\sqrt{17} + 20 + \frac{1}{5}\left[\frac{(-1+\sqrt{17})^5}{32} + 32 \right]$$

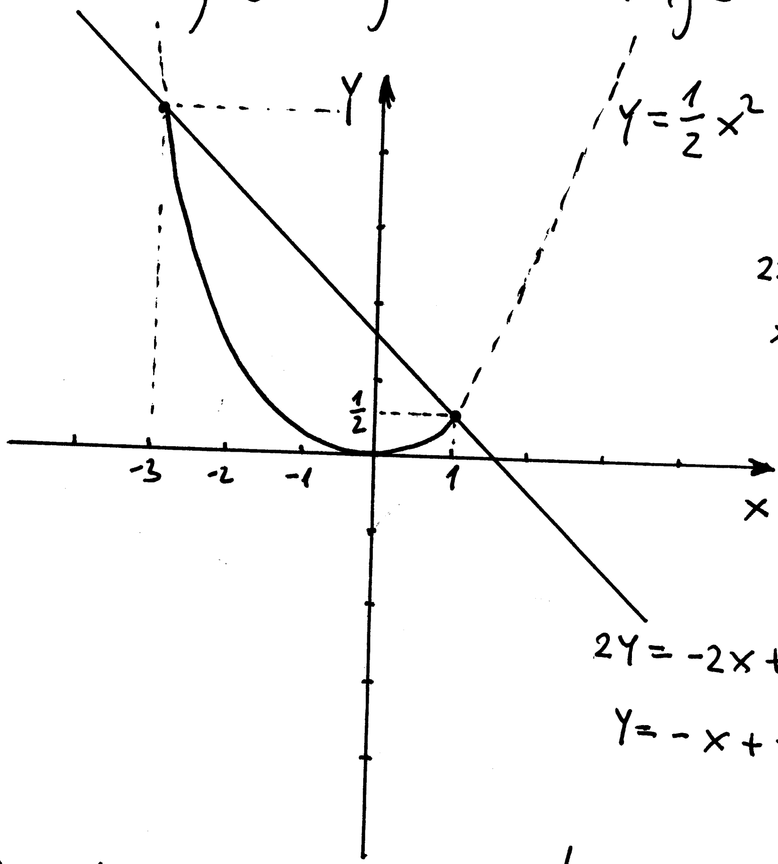
$$V_2 = \pi \int_0^{\frac{-1+\sqrt{17}}{2}} x^2\,dx = \pi \frac{1}{3} x^3 \Big|_0^{\frac{-1+\sqrt{17}}{2}} = \frac{\pi}{3}\left(\frac{1}{8}\sqrt{17}^3 - \frac{3}{8}\sqrt{17}^2 + \frac{3}{8}\sqrt{17} - \frac{1}{8} \right) = \frac{\pi}{24}\sqrt{17}^3 + \frac{\pi\sqrt{17}}{8} - \frac{13\pi}{6}$$

$$V_x = V_1 - V_2 = \frac{-\sqrt{17}^3 \pi}{24} - \frac{\sqrt{17}^3}{3} - \frac{\sqrt{17}\pi}{8} + 7\sqrt{17} + \frac{13\pi}{6} + 20 + \frac{1}{5}\left[\frac{(-1+\sqrt{17})^5}{32} + 32 \right]$$

tačno rješenje

(#) Figura u ravni ograničena linijama $2y = x^2$ i $2x + 2y - 3 = 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

Rj. Skicirajmo dvije date linije



$2y = -2x + 3$

$y = -x + \frac{3}{2}$

$2x + 2y - 3 = 0 \;/:2$

$x + y - \frac{3}{2} = 0$

$y = \frac{1}{2}x^2$

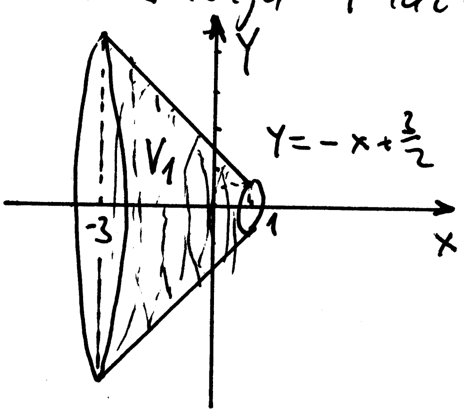
$$\begin{aligned} 2y &= x^2 \\ 2x + 2y - 3 &= 0 \\ \hline x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \end{aligned}$$

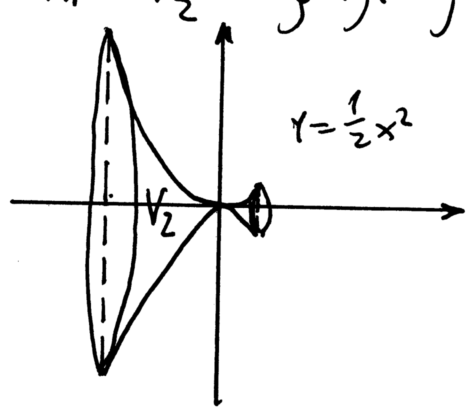
$x_1 = -3 \Rightarrow y_1 = \frac{9}{2}$

$x_2 = 1 \Rightarrow y_2 = \frac{1}{2}$

Prisjetimo se: $V_x = \pi \int_a^b y^2\,dx$ je zapremina tijela kada f-ja $y = f(x)$ rotira oko x-ose, za $a \leq x \leq b$

U našem slučaju imademo $V = V_1 - V_2$ gdje je





$$V_1 = \pi \int_{-3}^{1} \left(-x+\frac{3}{2}\right)^2 dx = \pi \int_{-3}^{1} \left(x^2 - 3x + \frac{9}{4}\right) dx =$$

$$= \frac{1}{3}x^3\Big|_{-3}^{1} - \frac{3}{2}x^2\Big|_{-3}^{1} + \frac{9}{4}x\Big|_{-3}^{1} = \ \dots = \frac{91}{3}\pi$$

integral u V_1 smo mogli izračunati i na drugi način

$$V_1 = \pi \int_{-3}^{1} \left(-x+\frac{3}{2}\right)^2 dx = \pi \int_{-3}^{1} (-1)^2 \left(x-\frac{3}{2}\right)^2 d\left(x-\frac{3}{2}\right) =$$

$$= \pi \cdot \frac{1}{3}\left(x-\frac{3}{2}\right)^3\Big|_{-3}^{1} = \ \dots = \frac{91\pi}{3}$$

$$V_2 = \pi \int_{-3}^{1} \left(\frac{1}{2}x^2\right)^2 dx = \frac{\pi}{4}\int_{-3}^{1} x^4\, dx = \frac{\pi}{4}\cdot\frac{1}{5}x^5\Big|_{-3}^{1} = \frac{\pi}{20}\left(1+243\right) =$$

$$= \frac{61}{5}\pi$$

$$V = V_1 - V_2 = \frac{91\pi}{3} - \frac{61\pi}{5} = \frac{272\pi}{15} = 18\frac{2}{15}\pi$$

tražena zapremina

(#) Naći ekstreme f-je $z = x + y + 4 + 4\sin x \sin y$.

Rj. $\frac{\partial z}{\partial x} = 1 + 4 \cdot \cos x \sin y$

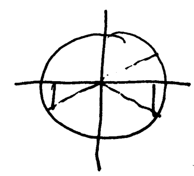
$\frac{\partial z}{\partial y} = 1 + 4 \sin x \cos y$

$1 + 4\cos x \sin y = 0$
$1 + 4\sin x \cos y = 0$

$\sin y \cos x = -\frac{1}{4}$ (a)
$\sin x \cos y = -\frac{1}{4}$ (b)

(a)+(b): $\sin y \cos x + \sin x \cos y = -\frac{1}{2}$

$\sin(y + x) = -\frac{1}{2}$



$y + x = \frac{7\pi}{6}$ ili $y + x = \frac{11\pi}{6}$

(a)-(b):

$\sin y \cos x - \sin x \cos y = 0$

$\sin(y - x) = 0$

$y - x = 0$ ili $y - x = \pi$

1° $x + y = \frac{7\pi}{6}$
$+\ -x + y = 0$

$2y = \frac{7\pi}{6}$

$y = \frac{7\pi}{12} \Rightarrow x = \frac{7\pi}{12}$

2° $x + y = \frac{7\pi}{6}$
$-x + y = \pi$

$2y = \frac{13\pi}{6}$

$y = \frac{13\pi}{12} \Rightarrow x = \frac{\pi}{12}$.

3° $x + y = \frac{11\pi}{6}$
$-x + y = 0$

$y = \frac{11\pi}{12} \Rightarrow$
$x = \frac{11\pi}{12}$

4° $y + x = \frac{11\pi}{6}$
$+\ y - x = \pi$

$y = \frac{17\pi}{12} \Rightarrow x = \frac{5\pi}{12}$

Stacionarne tačke su
$M_1\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right)$, $M_2\left(\frac{\pi}{12}, \frac{13\pi}{12}\right)$,
$M_3\left(\frac{11\pi}{12}, \frac{11\pi}{12}\right)$, $M_4\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right)$

$\frac{\partial^2 z}{\partial x^2} = -4 \sin x \sin y$

$\frac{\partial^2 z}{\partial x \partial y} = 4 \cos x \cos y$

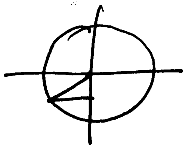
$\frac{\partial^2 z}{\partial y^2} = -4 \sin x \sin y$

$\cos(x+y) = \cos x \cos y - \sin x \sin y$ (I)
$\cos(x-y) = \cos x \cos y + \sin x \sin y$ (II)

(I) + (II): $\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$

(II) − (I): $\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

• Za $M_1\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right)$



$\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

$A = -4 \cdot \frac{1}{2}\left(\cos 0 - \cos\frac{7\pi}{6}\right) = -2\left(1 + \frac{\sqrt{3}}{2}\right) = -2 - \sqrt{3}$

$B = 4 \cdot \frac{1}{2}\left(\cos\frac{7\pi}{6} + \cos 0\right) = 2\left(-\frac{\sqrt{3}}{2} + 1\right) = -\sqrt{3} + 2$

$C = -2 - \sqrt{3}$

$D = AC - B^2 = (2+\sqrt{3})^2 - (2-\sqrt{3})^2 > 0 \Rightarrow$ f-ja ima ekstrem

$A < 0$ f-ja u tački M_1 ima maksimum

$z_{max}\left(\frac{7\pi}{12}, \frac{7\pi}{12}\right) = \frac{7\pi}{12} + \frac{7\pi}{12} + 4 + 2 + \sqrt{3} = 6 + \sqrt{3} + \frac{7\pi}{6}$ traženi ekstrem

• Za $M_2\left(\frac{\pi}{12}, \frac{13\pi}{12}\right)$

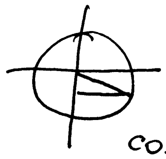
$A = -4 \cdot \frac{1}{2}\left(\cos(-\pi) - \cos\frac{7\pi}{6}\right) = -2\left(-1 + \frac{\sqrt{3}}{2}\right) = 2 - \sqrt{3}$

$B = 4 \cdot \frac{1}{2}\left(\cos\frac{7\pi}{6} + \cos(-\pi)\right) = 2\left(-\frac{\sqrt{3}}{2} - 1\right) = -\sqrt{3} - 2$

$C = 2 - \sqrt{3}$

$D = AC - B^2 = (2-\sqrt{3})^2 - (2+\sqrt{3})^2 > 0 \Rightarrow$ f-ja u tački M_2 nema ekstrema

• Za $M_3\left(\frac{11\pi}{12}, \frac{11\pi}{12}\right)$



$\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

$A = -4 \cdot \frac{1}{2}\left(\cos 0 - \cos\frac{11\pi}{6}\right) = -2\left(1 - \frac{\sqrt{3}}{2}\right) = -2 + \sqrt{3}$

$B = 4 \cdot \frac{1}{2}\left(\cos\frac{11\pi}{6} + \cos 0\right) = 2\left(\frac{\sqrt{3}}{2} + 1\right) = \sqrt{3} + 2$

$C = \sqrt{3} - 2$

$D = (\sqrt{3} - 2)^2 - (\sqrt{3} + 2)^2 < 0 \Rightarrow$ f-ja u tački M_3 nema ekstrem

• Za $M_4\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right)$

$A = -4 \cdot \frac{1}{2}\left(\cos(-\pi) - \cos\frac{11\pi}{6}\right) = -2\left(-1 - \frac{\sqrt{3}}{2}\right) = 2 + \sqrt{3}$

$B = 4 \cdot \frac{1}{2}\left(\cos\frac{11\pi}{6} + \cos(-\pi)\right) = 2\left(\frac{\sqrt{3}}{2} - 1\right) = \sqrt{3} - 2$

$C = 2 + \sqrt{3}$, $D = AC - B^2 = (2+\sqrt{3})^2 - (\sqrt{3} - 2)^2 > 0 \Rightarrow$ f-ja u tački M_4 ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}\left(\frac{5\pi}{12}, \frac{17\pi}{12}\right) = \frac{5\pi}{12} + \frac{17\pi}{12} + 4 + (-2 - \sqrt{3}) = 2 - \sqrt{3} + \frac{11\pi}{6}$ traženi ekstrem

(#) Odrediti ekstreme f-je $f(x,y)=xe^{y+x\sin y}$.

Rj: Odredimo parcijalne izvode

$$\frac{\partial f}{\partial x}=e^{y+x\sin y}+xe^{y+x\sin y}\cdot\sin y=e^{y+x\sin y}(1+x\sin y)$$

$$\frac{\partial f}{\partial y}=xe^{y+x\sin y}\cdot(1+x\cos y)$$

Da bi odredili stacionarne tačke trebamo riješiti sljedeći sistem

$$\frac{\begin{aligned}e^{y+x\sin y}(1+x\sin y)&=0\\ xe^{y+x\sin y}(1+x\cos y)&=0\end{aligned}}{}$$

primjetimo da je $e^{y+x\sin y}>0\ \forall x,y$

$/e^{y+x\sin y}$ obe jednačine

$$\frac{\begin{aligned}1+x\sin y&=0\\ x(1+x\cos y)&=0\end{aligned}}{}$$

primjetimo da x ne smije biti nula (u suprotnom iz prve jednakosti bi dobili $1=0$ # kontrad.)

$$\frac{\begin{aligned}1+x\sin y&=0\\ 1+x\cos y&=0\end{aligned}}{\sin y=\cos y}$$

$y=\frac{\pi}{4}+2k\pi,\ k\in\mathbb{Z}$

ili $y=\frac{5\pi}{4}+2k\pi,\ k\in\mathbb{Z}$

Za $y=\frac{\pi}{4}+2k\pi$ imamo

$$1+x\frac{\sqrt{2}}{2}=0$$

$$x=-\sqrt{2}$$

Za $y=\frac{5\pi}{4}+2k\pi$ imamo

$$1+x\left(-\frac{\sqrt{2}}{2}\right)=0$$

$$x=\sqrt{2}$$

Stacionarne tačke su

$M_k(-\sqrt{2};\frac{\pi}{4}+2k\pi)$

i $N_k(\sqrt{2};\frac{5\pi}{4}+2k\pi)$.

Odredimo druge parcijalne izvode

$$\frac{\partial^2 f}{\partial x^2} = e^{y+x\sin y}\cdot \sin y + e^{y+x\sin y}\sin y = 2e^{y+x\sin y}\sin y$$

$$\frac{\partial^2 f}{\partial x \partial y} = e^{y+x\sin y}\cdot(1+x\cos y)(1+x\sin y) + e^{y+x\sin y}x\cos y$$

$$\frac{\partial^2 f}{\partial y^2} = x e^{y+x\sin y}\cdot(1+x\cos y)(1+x\cos y) + x e^{y+x\sin y}(-x\sin y)$$

Za tačke $M_k(-\sqrt{2}, \frac{\pi}{4}+2k\pi)$ imamo $1+x\sin y=0$ i $1+x\cos y=0$

pa je $A=2e^{\frac{\pi}{4}-\sqrt{2}\cdot\frac{\sqrt{2}}{2}}\cdot\frac{\sqrt{2}}{2}=\sqrt{2}\,e^{\frac{\pi}{4}-1}$, $B=-e^{\frac{\pi}{4}-1}$, $C=-\sqrt{2}\,e^{\frac{\pi}{4}-1}$

$$D=\begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC-B^2 = -2e^{\frac{\pi}{2}-2} - e^{\frac{\pi}{2}-2} = -3e^{\frac{\pi}{2}-2} < 0$$

U tačkama $M_k(-\sqrt{2}; \frac{\pi}{4}+2k\pi)$ f-ja nema ekstrem.

Za tačke $N_k(\sqrt{2}, \frac{5\pi}{4}+2k\pi)$ imamo

$A=-\sqrt{2}\,e^{\frac{5\pi}{4}-1}$, $B=-e^{\frac{5\pi}{4}-1}$, $C=\sqrt{2}\,e^{\frac{5\pi}{4}-1}$

$$D=\begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC-B^2 = -2e^{\frac{5\pi}{2}-2} - e^{\frac{5\pi}{2}-2} < 0$$

U tačkama $N_k(\sqrt{2}, \frac{5\pi}{4}+2k\pi)$ f-ja nema ekstrem.

(Tačke $M_k(-\sqrt{2}; \frac{\pi}{4}+2k\pi)$ i $N_k(\sqrt{2}; \frac{5\pi}{4}+2k\pi)$ su sedlaste tačke).

Prvo izračunati integral $\int_0^\infty e^{-x}\sin(\alpha x)\,dx$ pa poslije toga dobijeni rezultat iskoristiti i koristeći metodu diferenciranja po parametru izračunati

$$G(\alpha)=\int_0^\infty x e^{-x}\cos(\alpha x)\,dx .$$

Rj.

$$I(\alpha)=\int_0^\infty e^{-x}\sin(\alpha x)\,dx=\left|\begin{matrix} u=e^{-x} & dv=\sin\alpha x\,dx \\ du=-e^{-x}dx & v=-\frac{1}{\alpha}\cos\alpha x\end{matrix}\right|=$$

$$=-\frac{1}{\alpha}e^{-x}\cos\alpha x\Big|_0^\infty-\frac{1}{\alpha}\int_0^\infty e^{-x}\cos(\alpha x)\,dx=\left|\begin{matrix} u=e^{-x} & dv=\cos\alpha x\,dx \\ du=-e^{-x} & v=\frac{1}{\alpha}\sin\alpha x\end{matrix}\right|$$

ovdje podrazumijevano da se računa $\lim_{A\to\infty}(e^{-x}\cos\alpha x)\Big|_0^A$

$$=\left(0+\frac{1}{\alpha}\cdot 1\right)-\frac{1}{\alpha^2}\underbrace{e^{-x}\sin\alpha x\Big|_0^\infty}_{0-0}-\frac{1}{\alpha^2}\int_0^\infty e^{-x}\sin\alpha x\,dx$$

ovdje podrazumijevam da se računa $\lim_{A\to\infty}(e^{-x}\sin\alpha x)\Big|_0^A$

$$\Rightarrow\quad I(\alpha)=\frac{1}{\alpha}-\frac{1}{\alpha^2}I(\alpha)$$

$$I(\alpha)+\frac{1}{\alpha^2}I(\alpha)=\frac{1}{\alpha}\quad\Rightarrow\quad\left(1+\frac{1}{\alpha^2}\right)I(\alpha)=\frac{1}{\alpha}$$

$$\frac{\alpha^2+1}{\alpha^2}I(\alpha)=\frac{1}{\alpha}\quad /\cdot\alpha$$

$$I(\alpha)=\frac{\alpha}{\alpha^2+1}$$

Označimo sa $F(\alpha)=\int_0^\infty e^{-x}\sin(\alpha x)\,dx=\frac{\alpha}{\alpha^2+1}$.

Kako je $\left(e^{-x}\sin(\alpha x)\right)'_\alpha=xe^{-x}\cos\alpha x$ i

$$\left(\frac{\alpha}{\alpha^2+1}\right)'_\alpha = \frac{1\cdot(\alpha^2+1)-\alpha\cdot 2\alpha}{(\alpha^2+1)^2} = \frac{1-\alpha^2}{(\alpha^2+1)^2}$$

To je

$$F'(\alpha) = \int_0^\infty x\, e^{-x} \cos\alpha x \, dx = \frac{1-\alpha^2}{(\alpha^2+1)^2}$$

Pa je

$$G(\alpha) = \int_0^\infty x\, e^{-x} \cos\alpha x \, dx = \frac{1-\alpha^2}{(\alpha^2+1)^2}$$

traženo rješenje

(#) Date su vrijednosti dva integrala $(\alpha>0)$

$$\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx=\frac{\pi}{2}e^{-\alpha} \quad i \quad \int_0^\infty \frac{\sin\alpha x}{x}\,dx=\frac{\pi}{2}.$$

Koristeći date jednakosti uz pomoć metode diferenciranja po parametru izračunati

$$\int_0^\infty \frac{\sin\alpha x}{x(1+x^2)}\,dx.$$

Rj: Označimo sa $F(\alpha)=\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx=\frac{\pi}{2}e^{-\alpha}$.

Kako je

$$\left(\frac{\cos\alpha x}{1+x^2}\right)'_\alpha=\frac{-x\sin\alpha x}{1+x^2} \quad i \quad \left(\frac{\pi}{2}e^{-\alpha}\right)'_\alpha=-\frac{\pi}{2}e^{-\alpha}$$

To je

$$F'_\alpha=\int_0^\infty \frac{-x\sin\alpha x}{1+x^2}\,dx=-\frac{\pi}{2}e^{-\alpha} \quad tj. \quad \int_0^\infty \frac{x\sin\alpha x}{1+x^2}\,dx=\frac{\pi}{2}e^{-\alpha}$$

Sad primjetimo da je

$$\frac{x\sin\alpha x}{1+x^2}=\frac{x^2\sin\alpha x^{\;+\sin\alpha x-\sin\alpha x}}{x(1+x^2)}=\frac{x^2\sin\alpha x+\sin\alpha x}{x(1+x^2)}-\frac{\sin\alpha x}{x(1+x^2)}=$$

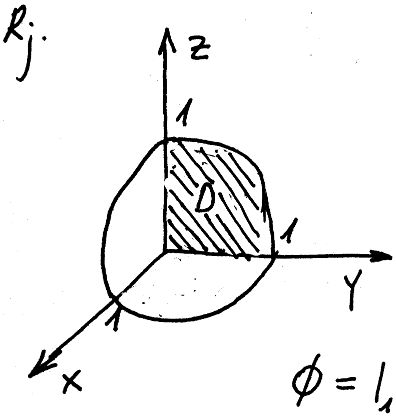
$$=\frac{\sin\alpha x\cdot(x^2+1)}{x\cdot(1+x^2)}-\frac{\sin\alpha x}{x(1+x^2)}=\frac{\sin\alpha x}{x}-\frac{\sin\alpha x}{x(1+x^2)}$$

Pa imamo

$$\int_0^\infty \frac{\sin\alpha x}{x(1+x^2)}\,dx=\overbrace{\int_0^\infty \frac{\sin\alpha x}{x}\,dx}^{\frac{\pi}{2}}-\overbrace{\int_0^\infty \frac{x\sin\alpha x}{1+x^2}\,dx}^{\frac{\pi}{2}e^{-\alpha}}=\frac{\pi}{2}\left(1-e^{-\alpha}\right)$$

(#) Naći fluks polja $\vec{v} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ kroz dio sfere $x^2+y^2+z^2=1$ u I oktantu.

Rj.



I način

$$\phi = \iint_S \vec{v}\cdot\vec{n}\, dS = \iint_S (V_x\cos\alpha + V_y\cos\beta + V_z\cos\gamma)\, dS$$

$$= \iint_S V_x\, dy\, dz + V_y\, dx\, dz + V_z\, dx\, dy$$

$$\phi = I_1 + I_2 + I_3 = \iint_S xy\, dy\, dz + \iint_S yz\, dx\, dz + \iint_S zx\, dx\, dy$$

Zbog simetrije $I_1 = I_2 = I_3$ pa je $\phi = 3I_1$. Računamo samo I_1

$$I_1 = \iint_S xy\, dy\, dz = +\iint_D \sqrt{1-(y^2+z^2)}\, y\, dy\, dz \quad \text{gdje je } D: y^2+z^2 \le 1,\ x\ge 0,\ y \ge 0$$

$x^2 = 1-(y^2+z^2)$

$x = \pm\sqrt{1-(y^2+z^2)}$

Vektor normale zaklapa ugao $\alpha \in (0, \frac{\pi}{2})$ sa x-osom. $\cos\alpha > 0$ (u I oktantu).

uzimamo + jer smo u prvom oktantu

Uvodimo polarne koordinate $y = r\cos\varphi$, $z = r\sin\varphi$

$$D': \begin{cases} 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le r \le 1 \\ dy\, dz = r\, d\varphi\, dr \end{cases} \qquad y^2+z^2 = r^2$$

$$I_1 = \iint_{D'} r\cos\varphi \sqrt{1-r^2}\cdot r\, d\varphi\, dr = \int_0^1 r^2\sqrt{1-r^2}\left[\int_0^{\frac{\pi}{2}} \cos\varphi\, d\varphi\right] dr = \int_0^1 r^2\sqrt{1-r^2}\cdot 1\, dr$$

$$= \left| \begin{matrix} r = \sin t \\ dr = \cos t\, dt \end{matrix} \right| = \int_0^1 \sin^2 t\sqrt{1-\sin^2 t}\cos t\, dt = \int_0^1 \sin^2 t\cos^2 t\, dt = \ldots = \frac{3\pi}{16}$$

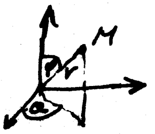
u prethodnom zadatku smo imali slično

II način: Kako je S zatvorena površ možemo primjeniti formulu Gauss-Ostrogradski.

$$\phi = \iint_S \vec{v}\cdot\vec{n}\, dS = \iiint_\Omega \operatorname{div}\vec{v}\, dx\, dy\, dz = \iiint_\Omega \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\, dy\, dz$$

U našem slučaju $\phi = \iiint_\Omega (x+y+z)\, dx\, dy\, dz$ gdje je $\Omega: \begin{cases} x^2+y^2+z^2 \le 1 \\ x \ge 0,\ y \ge 0 \\ z \ge 0 \end{cases}$

Uvodimo sferne koordinate



$x = r\sin\varphi\cos\theta$
$y = r\sin\varphi\sin\theta$
$z = r\cos\varphi$

$$\Rightarrow \Omega': \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi/2 \\ 0 \le \theta \le \pi/2 \end{cases}$$

$dx\, dy\, dz = r^2\sin\varphi\, d\varphi\, dr\, d\theta$

$$\phi = \iiint_{\Omega'} (r\sin\varphi\cos\theta + \ldots)\, r^2\sin\varphi \ldots = \ldots = \frac{3\pi}{16}$$

Izračunati tok (fluks) vektora $\vec{v} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ kroz sferu $x^2+y^2+z^2=R^2$.

Rj. $\vec{v} = (V_x, V_y, V_z) = (x^3, y^3, z^3)$

Tok vektorskog polja (kroz površ S) je površinski integral

$$\phi = \iint_S V_x\,dy\,dz + V_y\,dx\,dz + V_z\,dx\,dy$$

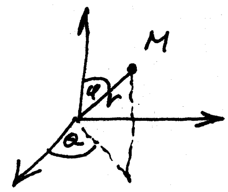
Kako je data zatvorena površina S to možemo upotrebiti formulu Gauss-Ostrogradski

$$\iint_S V_x\,dy\,dz + V_y\,dx\,dz + V_z\,dx\,dy = \iiint_\Omega \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\,dy\,dz$$

$\frac{\partial V_x}{\partial x} = 3x^2$, $\frac{\partial V_y}{\partial y} = 3y^2$, $\frac{\partial V_z}{\partial z} = 3z^2$, Ω oblast ograničena sferom $x^2+y^2+z^2=R^2$

$$\phi = \iiint_\Omega 3(x^2+y^2+z^2)\,dx\,dy\,dz \overset{(\Delta)}{=\!=}$$

uvedimo sferne koordinate



$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi$

$x^2+y^2+z^2 = r^2[\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi] = r^2$

$$\Omega': \begin{cases} 0 \le r \le R \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{cases}$$

$$\overset{(\Delta)}{=\!=} 3\iiint_{\Omega'} r^2 r^2 \sin\varphi\,dr\,d\varphi\,d\theta =$$

$$= 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\,d\varphi \int_0^R r^4\,dr = 3\int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi \frac{1}{5} r^5\Big|_0^R d\varphi = 3\frac{R^5}{5}\int_0^{2\pi} (-\cos\varphi)\Big|_0^{\pi} d\theta$$

$$= \frac{6R^5}{5}\pi\, \theta\Big|_0^{2\pi} = \frac{12R^5}{5}\pi$$ traženi tok vektora kroz sferu